TEMPERATURE-DEPENDENT VISCOELASTIC PROPERTIES OF UNIDIRECTIONAL COMPOSITE MATERIALS

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Keywords: *composite materials; temperature-dependent viscoelasticity; thermal expansion; constitutive equations; homogenization theory*

Abstract

The understanding of the viscoelastic behavior that accounts for the heterogeneous composite material's microstructure and the temperature-dependency of properties still requires further study and development. A homogenization-based method is used to study the property of time-temperature-dependent viscoelastic behavior of unidirectional fiber reinforced composites with thermorheological simple matrix, and a novel multi-scale method for analyzing the viscoelastic property of unidirectional composite materials is developed. The character of the viscoelastic relaxation law subject to variation in temperature is investigated, and the global effective viscoelastic relaxation modulus considering the temperature-dependency of properties was given. Numerical examples for a plate of unidirectional fiber reinforced composite material are presented. Through the analysis of the thermal strain of the plate, it is found that the time effect of the relaxation is small, so the coefficient of thermal expansion may be given by the instantaneous thermal deformation.

1 Introduction

Composite materials have been widely used in advanced industries for their inherent advantages, such as designable character, high specific strength and specific stiffness. Researches have physically shown that composite materials can exhibit viscoelastic behavior, particularly those containing polymers and viscoelasticity matrix [1-2].

The understanding of the viscoelastic behaviors for heterogeneous composite materials that accounts for their microstructures still requires further study and development. Therefore, it is very important to study micro-mechanical methods for predicting numerically or analytically the time-temperature-dependent viscoelastic properties of composite materials based on the heterogeneous microstructural details of composite materials.

Existing micromechanical methods for predicting the properties of materials include self-consistent theory [3], Eshelby-Mori-Tanaka's method [4,5], the cells model [6,7], homogenization method [8-12] and others. These methods have been successfully used to predict the elastic properties of composites [13]. The micromechanical methods have been employed to predict the viscoelastic properties of composite materials [14-19]. For the unidirectional fiber reinforced polymeric matrix composites, based on the micromechanical theory and the elastic-viscoelastic correspondence principle, Li and Weng [14] studied the attenuation laws with time of the five independent viscoelastic constants of composite materials. Liang and Du [15] used the Eshelby equivalent inclusion method to study the creep constitutive relations of particulate reinforced composites, and developed the variation laws for the moduli of materials with time, inclusion volume and load. Based on the homogenization methods, Liu et al. [16] predicted the viscoelastic properties of multi-layered materials and unidirectional fiber reinforced composites, and investigated the effect of the inclusion's volume fraction on the relaxation modulus. Chung et al. [17] proposed a micro/macro homogenization approach for viscoelastic creep analysis with dissipative correctors for hererogeneous woven-fabric layered materials.

Seiferta et al. [18] employed finite element method to predict the viscoelastic properties of an E-glass/vinylester plain weave, woven roving composite material at three different temperatures, and compared them with experimental data. These studies gave the viscoelastic laws at constant temperatures, however the properties of composite materials are temperature-dependent. Though Seiferta et al. [18] gave the viscoelastic properties of composite material for three different temperatures, the influence of the varying temperature history on the property of composite materials wasn't considered. The study on the viscoelastic laws under varying temperature states has rarely been reported. In our previous work [19], the viscoelastic laws under a very particularly varying temperature states, in which the temperature has a jump at a specific time (e.g, at the beginning) and then keeps constant, were studied. The viscoelastic properties under general varying temperature stage should be investigated.

Zhang et al. [20] systemically studied the viscoelastic properties of single phase materials under time-dependent temperature change, and expressed the viscoelastic constitutive equations under varying temperature states in the same form as that under constant temperature states. For composite materials, although the constitutive equations under a time-independent temperature state can be expressed as the same form as that of a single phase, it is still unknown if the constitutive equation under time-dependent varying temperature state can be expressed as a similar form.

The main purpose of this paper is to develop a novel multi-scale method for analyzing the viscoelastic properties of composite materials, and to investigate the characteristics of the viscoelastic relaxation law under time-dependent temperature changes. Based on the homogenization theory, the multi-scale analysis methods of the viscoelastic properties and the effective thermal stress relaxation laws are studied. Numerical examples are presented in the end of the paper.

2 Time-Temperature-Dependent Constitutive Equation of Unidirectional Composite Materials

In this paper, we will investigate the time-temperature-dependent constitutive equation of unidirectional fiber reinforced composite materials. The fibers are elastic materials and the matrix is a thermorheological simple material. The temperature considered here is time-dependent.

2.1 Viscoelastic Constitutive Equation

Owing to the heterogeneity of unidirectional fiber reinforced composites, the constitutive equation is different from position to position. Denote the domains occupied by the matrix and fiber as Ω_m and Ω_f , respectively, then the total domain of the composites is the sum of this two domains.

$$\Omega = \Omega_f + \Omega_m \tag{1}$$

In the fiber phase domain, the material exhibits elastic property, and the constitutive equation can be expressed as

$$\sigma_{ij}(x,t) = E_{ijkl}\varepsilon_{kl}(x,t), \quad x \in \Omega_f \tag{2}$$

For the thermorheological simple materials, through the horizontal function $\chi(T)$, the curve of relaxation at different temperatures can be brought into a single one on the reduced-time scale ξ ($\xi = t\chi(T)$), thereby forming the basis of the 'time-temperature superposition principle'. Experimental data of many polymers indicated the existence of such a shift factor α_T, for that

$$\log\chi(T) = -\log\alpha_T \tag{3}$$

Based on the William-Landel-Ferry equation (WLF), one has:

$$\log\alpha_T = -\frac{C_1(T-T_r)}{C_2+T-T_r} \tag{4}$$

Where, the parameter C_1 and C_2 are related to the free volume of the polymer and are dependent on the chosen reference temperature T_r.

For a temperature change history T(t) relative to the reference temperature T_r, the reduced-time scale ξ is given by

$$\xi = \int_0^t \exp[C_1T(\eta)/(C_2+T(\eta))]\,d\eta \tag{5}$$

Thus, the constitutive equation of materials in the matrix domain can be expressed as:

$$\sigma_{ij}(t) = \int_0^t G_{ijkl}(\xi-\xi')[\frac{d\varepsilon_{kl}(\tau)}{d\tau} - \alpha_{kl}\frac{dT(\tau)}{d\tau}]d\tau \tag{6}$$

where,

$$\xi' = \int_0^\tau \exp[C_1T(\eta)/(C_2+T(\eta))]\,d\eta \tag{7}$$

By the integral transition, and t, τ being replaced by ξ , ξ' respectively, then Eq.(5) is rewritten as:

$$\sigma_{ij}(t)=\int_0^{\xi} G_{ijkl}(\xi-\xi')[\frac{d\varepsilon_{kl}(\xi')}{d\xi'}-\alpha_{kl}\frac{dT(\xi')}{d\xi'}]d\xi' \quad (8)$$

If we define the relaxation modulus of the fiber material as

$$G_{ijkl}^{(f)}(\xi)=E_{ijkl} \quad (9)$$

Then, the constitutive equation of the fiber can be expressed as a similar form as Eq.(8).

$$\sigma_{ij}(t)=\int_0^{\xi} G_{ijkl}^{f}(\xi-\xi')[\frac{d\varepsilon_{kl}(\xi')}{d\xi'}-\alpha_{kl}\frac{dT(\xi')}{d\xi'}]d\xi' \quad (10)$$

2.2 Effective Time-Temperature-Dependent Viscoelastic Properties of Composites

For a unidirectional fiber reinforced composite material with elastic fibers distributed periodically in a thermorheologically simple viscoelastic matrix, the viscoelastic governing equation of the material under the action of body forces, tractions and temperature increments, can be expressed as

$$\int_{\Omega}\{\sigma_{ij}(t)\}\frac{\partial v_i}{\partial x_j}\mathrm{d}\Omega-\int_{\Omega} f_i\, v_i\mathrm{d}\Omega-\int_{\Gamma_t} t_i v_i \mathrm{ds}=0, \quad (11)$$
$$\forall v\in \mathrm{V}=\{\boldsymbol{v}(\boldsymbol{x})\,|\,\boldsymbol{x}\in\Omega,\quad \boldsymbol{v}|\Gamma_d=0\}$$

Substituting Eq.(10) into Eq.(11), we then have:

$$\int_{\Omega}\int_0^{\xi} G_{ijkl}(\boldsymbol{x},\xi-\xi')\left[\frac{\partial^2 u_k(\boldsymbol{x},\xi')}{\partial x_l\partial\xi'}d\tau-\alpha_{kl}\frac{\partial T(\boldsymbol{x},\xi')}{\partial\xi'}\right]d\xi'$$
$$\frac{\partial v_i}{\partial x_j}\mathrm{d}\Omega-\int_{\Omega} f_i v_i\mathrm{d}\Omega-\int_{\Gamma_t} t_i v_i\mathrm{ds}=0,\qquad \forall v\in \mathrm{V} \quad (12)$$

Laplace transformation of Eq. (12) yields

$$\int_{\Omega}\int_0^{\xi}\tilde{G}_{ijkl}(\boldsymbol{x},s)\left[\frac{\partial^2\tilde{u}_k(\boldsymbol{x},\xi')}{\partial x_l\partial\xi'}d\tau-\alpha_{kl}\tilde{T}(\boldsymbol{x},s)\right]d\xi'$$
$$\frac{\partial v_i}{\partial x_j}\mathrm{d}\Omega-\int_{\Omega} f_i v_i\mathrm{d}\Omega-\int_{\Gamma_t} t_i v_i\mathrm{ds}=0,\qquad \forall v\in \mathrm{V} \quad (13)$$

Based on the idea of homogenization theory [8,11,12], displacements can be expressed as a double-scale asymptotic series:

$$u_i(\boldsymbol{x},t)=u_i^0(\boldsymbol{x},t)+\varepsilon u_i^1(\boldsymbol{x},\mathbf{y},t)+\varepsilon^2 u_i^2(\boldsymbol{x},\mathbf{y},t)+\cdots \quad (14)$$

Substituting Eq. (14) into Eq. (13), equating the terms with the same power of ε yields:

$$\int_{\Omega} s\tilde{G}_{ijkl}[(\frac{\partial\tilde{u}_k^0}{\partial x_l}+\frac{\partial\tilde{u}_k^1}{\partial y_l})\frac{\partial v_i}{\partial y_j}+\frac{\partial\tilde{u}_k^0}{\partial y_l}\frac{\partial v_i}{\partial x_j}$$
$$-\alpha_{kl}\tilde{T}\frac{\partial v_i}{\partial y_j}]\mathrm{d}\Omega=0,\quad \forall\boldsymbol{v}(x,y)\in V_{\Omega XY} \quad (15)$$

$$\int_{\Omega} s\tilde{G}_{ijkl}[(\frac{\partial\tilde{u}_k^0}{\partial x_l}+\frac{\partial\tilde{u}_k^1}{\partial y_l})\frac{\partial v_i}{\partial x_j}+(\frac{\partial\tilde{u}_k^1}{\partial x_l}+\frac{\partial\tilde{u}_k^2}{\partial y_l})\frac{\partial v_i}{\partial y_j}$$
$$-\alpha_{kl}\tilde{T}\frac{\partial v_i}{\partial x_j}]\mathrm{d}\Omega-\int_{\Omega}\tilde{f}_i\, v_i\mathrm{d}\Omega-\int_{\Gamma_t}\tilde{t}_i v_i\mathrm{d}s=0, \quad (16)$$
$$\forall\boldsymbol{v}(x,\mathrm{y})\in V_{\Omega XY}$$

Where,

$$\forall\boldsymbol{v}(x,y)\in V_{\Omega\times\mathrm{Y}}=\{\boldsymbol{v}(\boldsymbol{x},\boldsymbol{y})\,|\,(\boldsymbol{x},\boldsymbol{y})\in\Omega\times\mathrm{Y},$$
$$|\ \boldsymbol{v}(\boldsymbol{x},\boldsymbol{y}+Y)=\boldsymbol{v}(\boldsymbol{x},\boldsymbol{y}),\quad \boldsymbol{v}|\Gamma_d=0\}$$

Taking the limit $\varepsilon\to 0$ and considering the periodic characters of the functions, Eqs. (15) and (16) become

$$\int_{\mathrm{Y}} s\tilde{G}_{ijkl}\left[(\frac{\partial\tilde{u}_k^0}{\partial x_l}+\frac{\partial\tilde{u}_k^1}{\partial y_l})-\tilde{G}_{ijkl}\alpha_{kl}\tilde{T}\right]\frac{\partial v_i}{\partial y_j}\mathrm{d}Y=0,$$
$$\forall\boldsymbol{v}(\mathrm{y})\in V_{\mathrm{Y}}=\{\boldsymbol{v}(\boldsymbol{y})\,|\,\boldsymbol{y}\in\mathrm{Y},|\ \boldsymbol{v}(\boldsymbol{y}+\mathrm{Y})=\boldsymbol{v}(\boldsymbol{y})\} \quad (17)$$

$$\int_{\Omega}\{\frac{1}{|Y|}\int_{Y} s\tilde{G}_{ijkl}[(\frac{\partial\tilde{u}_k^0}{\partial x_l}+\frac{\partial\tilde{u}_k^1}{\partial y_l}-\alpha_{kl}\tilde{T})\frac{\partial v_i}{\partial x_j}]\mathrm{d}\boldsymbol{y}\}\mathrm{d}\Omega$$
$$-\int_{\Omega}\tilde{f}_i v_i\mathrm{d}\Omega-\int_{\Gamma_t}\tilde{t}_i v_i\mathrm{ds}=0,\quad \forall\boldsymbol{v}(x,y)\in V_{\Omega XY} \quad (18)$$

Eq. (18) is linear about $\tilde{u}_k^{\,1}$, whose solution can be expressed as:

$$\tilde{u}_k^{\,1}=\tilde{u}_k^{\,1T}+\tilde{u}_k^{\,1u} \quad (19)$$

where, $\tilde{u}_k^{\,1T}$ and $\tilde{u}_k^{\,1u}$ are respectively the solutions of the following two equations:

$$\int_{\mathrm{Y}} s\tilde{G}_{ijkl}\left[\frac{\partial\tilde{u}_k^{1\theta}}{\partial y_l}-\alpha_{kl}\tilde{T}\right]\frac{\partial v_i}{\partial y_j}\mathrm{d}\boldsymbol{y}=0,\forall\boldsymbol{v}(\boldsymbol{y})\in\mathrm{V}_Y \quad (20)$$

$$\int_{\mathrm{Y}} s\tilde{G}_{ijkl}\left[\frac{\partial\tilde{u}_k^{1u}}{\partial y_l}+\frac{\partial\tilde{u}_k^0}{\partial x_l}\right]\frac{\partial v_i}{\partial y_j}\mathrm{d}\boldsymbol{y}=0,\forall\boldsymbol{v}(\boldsymbol{y})\in\mathrm{V}_Y \quad (21)$$

Owing to the linearity of Eqs.(20) and (21), Eq.(19) can be expressed as:

$$\tilde{u}_i^1=-\tilde{\chi}_i^{kl}\left(\partial\tilde{u}_k^0/\partial x_l\right)+\tilde{\Psi}_i\tilde{T},\quad i=1,2,3 \quad (22)$$

Where, the generalized functions $\tilde{\chi}_i^{kl}(\boldsymbol{y},s)$ and $\tilde{\Psi}_i(\boldsymbol{y},s)$ are respectively the solutions of the following two equations:

$$\int_{\mathrm{Y}}(\tilde{G}_{ijmp}-\tilde{G}_{ijkl}\frac{\partial\chi_k^{mp}}{\partial y_l})\frac{\partial v_i}{\partial y_j}\mathrm{d}Y=0,\forall\boldsymbol{v}(\boldsymbol{x},\boldsymbol{y})\in\mathrm{V}_Y \tag{23}$$

$$\int_{\mathrm{Y}}s\tilde{G}_{ijkl}(\frac{\partial\tilde{\Psi}_k}{\partial y_q}-\alpha_{kl})\frac{\partial v_i}{\partial y_j}\mathrm{d}Y=0,\forall\boldsymbol{v}(\boldsymbol{x},\boldsymbol{y})\in\mathrm{V}_Y \tag{24}$$

Introducing Eq.(22) into Eq. (18) yields

$$\begin{aligned}&\int_\Omega s\left[\tilde{G}_{ijkl}^{\mathrm{H}}\frac{\partial\tilde{u}_k^0}{\partial x_l}-\tilde{\beta}_{ij}^H\tilde{T}\right]\frac{\partial v_i}{\partial x_j}\mathrm{d}\Omega-\int_\Omega\tilde{f}_iv_i\mathrm{d}\Omega\\&-\int_{\Gamma_t}\tilde{t}_iv_i\mathrm{d}\Gamma=0,\qquad\forall\boldsymbol{v}(\boldsymbol{x})\in\mathrm{V}_\Omega\end{aligned} \tag{25}$$

where

$$\tilde{G}_{ijkl}{}^{\mathrm{H}}=\frac{1}{|\mathrm{Y}|}\int_{\mathrm{Y}}[\tilde{G}_{ijkl}-\tilde{G}_{ijmn}\frac{\partial\tilde{\chi}_m^{kl}}{\partial y_n}]\mathrm{d}Y \tag{26}$$

$$\tilde{\beta}_{ij}{}^{\mathrm{H}}=\frac{1}{|\mathrm{Y}|}\int_{\mathrm{Y}}[\tilde{G}_{ijkl}\alpha_{kl}-\tilde{G}_{ijkl}\frac{\partial\tilde{\Psi}_k}{\partial y_l}]\mathrm{d}Y \tag{27}$$

Eq.(25) is similar to Eq.(12), and $\tilde{\beta}_{ij}^{\mathrm{H}}$ has the same role as $\tilde{G}_{ijkl}\alpha_{kl}$ in Eq. (12). Therefore, $\tilde{G}_{ijkl}^H$ represents the Laplace transformatin of the effective viscoelastic relaxation modulus of composites, and $\beta_{ij}{}^{\mathrm{H}}$ is defined as the effective time-dependent thermal relaxation modulus (ETTRM).

ETTRM can be expressed in terms of $\tilde{\chi}_i^{kl}(\boldsymbol{y},s)$ as

$$\tilde{\beta}_{ij}{}^{\mathrm{H}}=\frac{1}{|Y|}\int_Y[\tilde{G}_{ijkl}\alpha_{kl}-\tilde{G}_{mnkl}\alpha_{kl}\frac{\partial\tilde{\chi}_m^{ij}}{\partial y_n}]\mathrm{d}Y \tag{28}$$

In fact, $\tilde{\chi}_i^{kl}(\boldsymbol{y},s)\in\mathrm{V}_{\mathrm{Y}}$, thus,

$$\int_Y[\tilde{G}_{mnkl}\alpha_{kl}-\tilde{G}_{mnkl}\frac{\partial\tilde{\Psi}_k}{\partial y_l}]\frac{\partial\tilde{\chi}_m^{ij}}{\partial y_n}\mathrm{d}Y=0 .$$

Considering this fact, the following equations can be obtained:

$$\tilde{\beta}_{ij}{}^{\mathrm{H}}=\frac{1}{|Y|}\int_Y[\tilde{G}_{ijkl}\alpha_{kl}-\tilde{G}_{mnkl}\alpha_{kl}\frac{\partial\tilde{\chi}_m^{ij}}{\partial y_n}]\mathrm{d}Y \tag{29}$$

Define macroscopic stress $\sigma_{ij}^{\mathrm{H}}(t)$ by:

$$\tilde{\sigma}_{ij}^{\mathrm{H}}=s\tilde{G}_{ijkl}^{\mathrm{H}}\left(\partial\tilde{u}_k^0/\partial x_l\right)-s\tilde{\beta}_{ij}^{\mathrm{H}}\tilde{T} \tag{30}$$

The inverse Laplace transformation yields:

$$\begin{aligned}\sigma_{ij}^{\mathrm{H}}(t)=&\int_0^{\xi(t)}G_{ijkl}^{\mathrm{H}}(\xi-\xi')\frac{\partial}{\partial\xi'}\frac{\partial u_k^0(x,\xi')}{\partial x_l}d\tau\\&-\int_0^{\xi(t)}\beta_{ij}^{\mathrm{H}}(\xi-\xi')\frac{\partial}{\partial\xi'}\frac{\partial T(x,\xi')}{\partial x_l}d\xi'\end{aligned} \tag{31}$$

Eq. (31) represents the effective viscoelastic constitutive equation of composite materials, which includes the thermal stress relaxation.

The stress and its Laplace transformation at any position in the domain of composite materials can be expressed as:

$$\sigma_{ij}(\boldsymbol{x},t)=\sigma_{ij}^0(\boldsymbol{x},t)+\varepsilon\sigma_{ij}^1(\boldsymbol{x},\mathrm{y},t)+\cdots \tag{32}$$

$$\tilde{\sigma}_{ij}(\boldsymbol{x},s)=\tilde{\sigma}_{ij}^0(\boldsymbol{x},s)+\varepsilon\tilde{\sigma}_{ij}^1(\boldsymbol{x},\mathrm{y},s)+\cdots \tag{33}$$

where,

$$\tilde{\sigma}_{ij}^0(\boldsymbol{x},s)=s\tilde{G}_{ijmn}\left(\frac{\partial\tilde{u}_m^{(0)}}{\partial x_n}-\frac{\partial u_m^{(1)}}{\partial y_n}\right)-s\tilde{G}_{ijmn}\alpha_{mn}\tilde{T} \tag{34}$$

$$\tilde{\sigma}_{ij}^{(k)}(\boldsymbol{x},s)=s\tilde{G}_{ijmn}\left(\frac{\partial\tilde{u}_m^{(k)}}{\partial y_n}-\frac{\partial u_m^{(k+1)}}{\partial y_n}\right),k=1,2,\cdots \tag{35}$$

Substitution of Eq. (22) into Eq. (34) yields

$$\begin{aligned}\tilde{\sigma}_{ij}^0(s)=&s\left(\tilde{G}_{ijkl}-\tilde{G}_{ijmn}\frac{\partial\tilde{\chi}_m^{kl}}{\partial y_n}\right)\frac{\partial\tilde{u}_k^0}{\partial x_l}\\&+s\tilde{G}_{ijmn}\left(\frac{\partial\tilde{\Psi}_m}{\partial y_n}-\alpha_{mn}\right)\tilde{T}\end{aligned} \tag{36}$$

Taking the volumetric average of the equation above over the base cell, we obtain:

$$\left\langle\tilde{\sigma}_{ij}^0(\mathrm{x},\mathrm{s})\right\rangle=\tilde{\sigma}_{ij}^{\mathrm{H}}(\mathrm{x},s) \tag{37}$$

which means that the macroscopic stress is the mean value of the first approximate stress over the base cell.

In this section, the viscoelastic constitutive equation of composite materials is presented. A homogenization-based multi-scale method is developed for predicting the effective relaxation modulus and ETTRM, and solving viscoelastic problem of composite material structures. The main schemes of this method are summarized as follows:

1) Obtain the generalized displace function $\chi_i^{kl}(\mathrm{y})$ by solving Eq. (28);

2) Calculate the effective relaxation modulus and ETSRM by inversely transforming Eqs. (26) and (28);

3) Solve Eq. (25) to get the macroscopic displace in the transformed space;

4) The stress which shows the local heterogeneous influence is obtained from the inverse transformation of Eq. (36)

5) The macroscopic stress and the constitutive equation are obtained by Eq. (31).

2.3 Thermal Expansion Property

For the conventional materials, if $T(t)$ is the history of temperature, the strain will be given by

$$\varepsilon_{ij}(t)=\int_{0^-}^{t}\alpha_{ij}\frac{dT(\tau)}{d\tau}d\tau \tag{38}$$

α_{ij} represents the thermal expansion coefficient and is a constant under constant temperature. So the thermal strain completes instantaneously under the uniformly increasing temperature.

Then the thermal strain of unidirectional composites can be expressed in a similar form under uniformly increasing temperature:

$$\varepsilon_{ij}(t)=\int_{0^-}^{t}\alpha_{ij}(t-\tau)\frac{dT(\tau)}{d\tau}d\tau \tag{39}$$

If the effective global stress equals to zero, the Laplace transform of the constitutive equation (30) is given by

$$\tilde{\varepsilon}_{kl}^{\mathrm{T}}=\left[\tilde{G}_{ijkl}^{\mathrm{H}}\right]^{-1}\tilde{\beta}_{kl}^{\mathrm{H}}\tilde{T} \tag{40}$$

If $\hat{\alpha}_{\mathrm{ij}}^{\mathrm{H}}(\xi(t))$ is defined by

$$\tilde{\hat{\alpha}}_{kl}^{\mathrm{H}}(s)=\frac{1}{s}\left[\tilde{G}_{ijkl}^{\mathrm{H}}\right]^{-1}\tilde{\beta}_{kl}^{\mathrm{H}} \tag{41}$$

Then the inverse Laplace transform is given by

$$\hat{\alpha}_{ij}^{\mathrm{H}}(\xi(t))=\mathrm{Re}v\left\{\frac{1}{s}\left[\tilde{G}_{ijkl}^{\mathrm{H}}\right]^{-1}\tilde{\beta}_{kl}^{\mathrm{H}}\right\} \tag{42}$$

$\mathrm{Re}v(f(s))$ represents the inverse Laplace transform for the function $f(s)$. Then the thermal strain related with temperature is given by

$$\varepsilon_{ij}(t)=\int_{0^-}^{t}\hat{\alpha}_{ij}^{H}(t-\tau)\frac{dT(\tau)}{d\tau}d\tau \tag{43}$$

That equation illustrates the thermal strain of unidirectional composites can not complete instantaneously. If $\hat{\alpha}_{\mathrm{ij}}^{\mathrm{H}}(\mathrm{t})$ is defined as the effective thermal expansion coefficient of unidirectional composites, it is dependent on temperature, and is called as the effective thermal expansion coefficient at final temperature.

3 Thermal Deformation Analysis of a Plate of Unidirectional Fiber Reinforced Composites

3.1 Problem Description

As an example, the viscoelastic thermal deformation of a unidirectional fiber reinforced composite plate is considered. The four edges of the plate are fixed as shown in Fig.1. The unidirectional fiber reinforced composites contain a kind of carbon fibers T300 embedded in a viscoelastic matrix. The matrix is a kind of resin called ED-6, and its volume deformation is elastic and shear deformation follows the three-parameter solid model as shown in Fig.2.

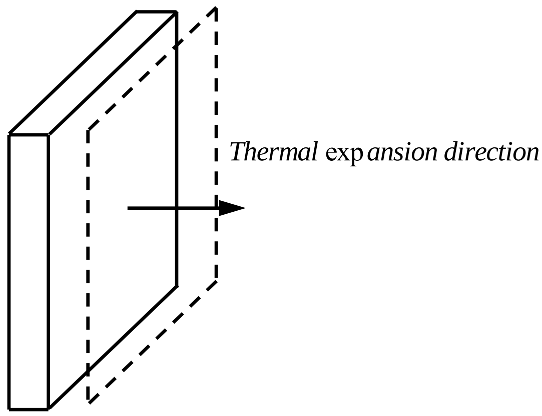

Fig.1 Thermal expansion of a plate

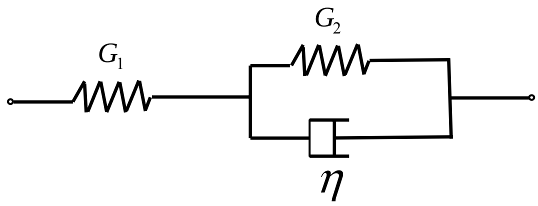

Fig.2 Three-parameter solid model

The relationship of the stress and the strain in the direction of thickness is given by

$$\sigma_{33}^{\mathrm{H}}(t)=\int_{0}^{\xi}G_{3333}^{\mathrm{H}}(\xi-\xi')\frac{\partial\varepsilon_{33}^{\mathrm{H}}(\xi')}{\partial\xi'} \\ -\beta_{33}^{\mathrm{H}}(\xi-\xi')\frac{dT(\xi')}{d\xi'}]d\xi'=0 \tag{44}$$

For the Laplace transform, it is written as

$$\tilde{\varepsilon}_{33}^{H}(s)=\left(\tilde{\beta}_{33}^{H}(s)/\tilde{G}_{3333}^{H}(s)\right)\tilde{T}(s) \tag{45}$$

The transform in the equation is for the reduced-time scale ξ. Based on the results under the constant temperature, the parameters in Eq.(45) are given by following equations and Table 1.

$$\tilde{G}_{3333}^{H}(s)=\frac{q_0^G}{s}+\frac{q^G}{s+p^G}$$
$$\tilde{\beta}_{33}^{H}(s)=\frac{q_0^\beta}{s}+\frac{q^\beta}{s+p^\beta} \tag{46}$$

Table 1. Parameters of the relaxation modulus under the reference temperature

	q_0(GPa)	q(GPa)	P(h^{-1})
G^{33}	71.1473	1.5109	0.0148
β^{33}	6.829×10^{-4}	0.0509×10^{-4}	0.013659

3.2 Laplace Transform of the History of Temperature

It is difficult to get the Laplace transform of the history of temperature. For the simplification, $T(t)$ is approximated as

$$T(t)=T_1H(t)+\sum_{i=1}^{n-1}\Delta T_i H(t-t_i) \tag{47}$$

Where, $H(t)$ is the Heaviside function. The approximated history of the temperature is shown in Fig.3. Then, the variation of the temperature with reduced-time scale is given by

$$[T(\xi)]=T_1H(\xi)+\sum_{k=2}^{i}(T_k-T_{k-1})H(\xi-\xi_{k-1}) \tag{48}$$

and the corresponding reduced-time scale becomes

$$\begin{aligned}&\xi_1=\int_0^{t_1}\chi[T(\tau)]d\tau=t_1\chi(T_1)\\&\xi_i=\int_0^{t_i}\chi[T(\tau)]d\tau=t_1\chi(T_1)+\sum_{k=2}^{i}(t_k-t_{k-1})\chi(T_k)\\&i=2,3,\cdots n-1\end{aligned} \tag{49}$$

The Laplace transform of the temperature and the strain can be expressed as

$$\left[\tilde{T}(\xi)\right]_1=\frac{T_1}{s},\left[\tilde{T}(\xi)\right]_i=\frac{T_1}{s}+\frac{1}{s}\sum_{k=2}^{i}(T_k-T_{k-1})e^{-\xi_{i-1}s} \tag{50}$$
$$i=2,3,\cdots,n$$

$$\varepsilon_{33}^{H}(\xi)=M_1+M_2*e^{-c_1\xi}+M_3*e^{-c_2\xi} \tag{51}$$

Where,

$$\begin{aligned}&M_1(1)=T_0*a\,,\ M_1(i)=\left(T_1+\sum_{k=2}^{i}(T_k-T_{k-1})\right)*a\\&M_2(1)=T_0*b\,,\ M_2(i)=\left(T_1+\sum_{k=2}^{i}(T_k-T_{k-1})e^{c_1\xi_{k-1}}\right)*b\\&M_3(1)=T_0*c\,,\ M_3(i)=\left(T_1+\sum_{k=2}^{i}(T_k-T_{k-1})e^{c_2\xi_{k-1}}\right)*c\end{aligned} \tag{52}$$

$$\begin{aligned}&b=A_1\left[\frac{(-p^\beta+p^G)(-p^\beta+A_3)}{(-p^\beta)(-p^\beta+A_2)}\right]\\&c=A_1\left[\frac{(-A_2+p^G)(-A_2+A_3)}{(-A_2)(-A_2+p^\beta)}\right]\\&a=A_1\left(\frac{p^GA_3}{p^\beta A_2}\right),\ c_1=p^\beta,\ c_2=A_2=\frac{q_0^Gp^G}{q_0^G+q^G}\end{aligned} \tag{53}$$

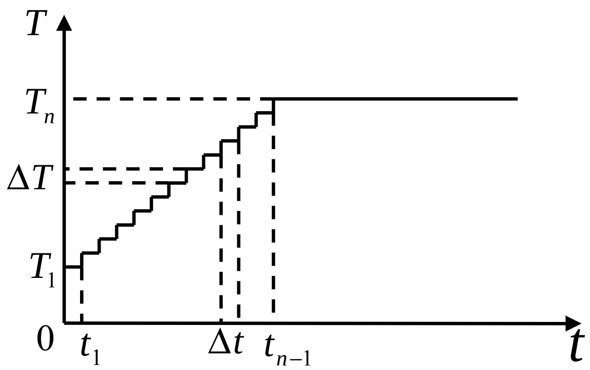

Fig.3 Uniformly increasing temperature

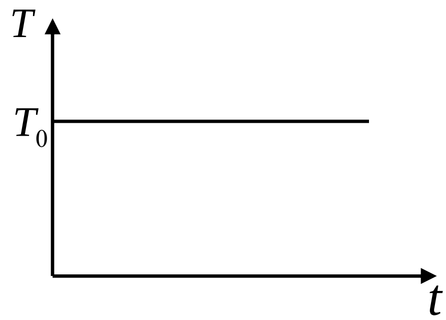

Fig.4 Instantaneously increasing temperature

3.3 Analysis of the results

Firstly, the thermal strain of the plate induced by a jump of temperature at the beginning are calculated. Fig.5 shows the strain of the plate with the jump of 20℃.

Then, the thermal strain of the plate induced by a uniformly increasing temperature at the beginning stage of time with a upper limit of temperature of 20 ℃ (as shown in Fig. 3) are calculated. Fig. 6 and Fig. 7 show the strain histories with increasing rates of temperature of 10℃/h, 20℃/h, and 40℃/h. The variation of the strain with time at the last step of temperature change is shown in Fig.8.

Through the above curves, we found that when the temperature increased to 20℃ uniformly, the increment of the strain is more than that under the instantaneously increasing to 20℃.The increment for the former is $\Delta\varepsilon = 1.4456e-4$, while it is for the latter is $\Delta\varepsilon = 2.593e-6$.

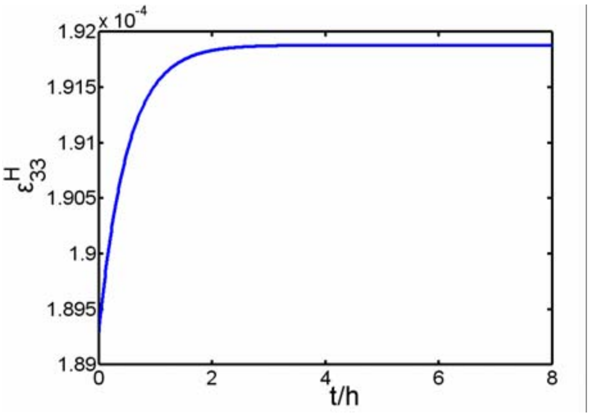

Fig.5 Strain curve under instantaneously increasing temperature

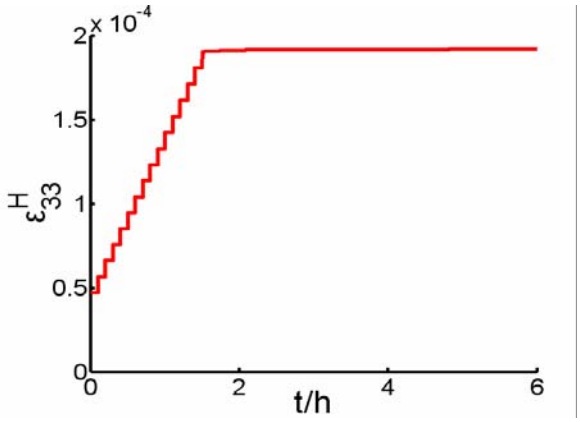

Fig.6 Strain curve under uniformly increasing temperature

Fig.7 shows the thermal strain that under the various rate of temperature. The relaxation is quick as the rate of the temperature increasing, and the final values of the strain are the same as the curves tend to constant.

When the temperature increased uniformly step by step, the relaxation rate is so high that the transient process is very quick. That is, the time effect of the relaxation is small, so the coefficient of the thermal expansion may be given by the transient thermal deformation. That is

$$\alpha(T(t),t) = \alpha(T(t)) \tag{54}$$

Thus, the thermal strain is given by

$$\varepsilon(T(t),t) = \int_0^t \alpha(T(\tau)) \frac{\partial T(\tau)}{\partial \tau} d\tau \tag{55}$$

And the thermal stress is

$$\begin{aligned}\sigma(T(t),t) &= \int_0^t \hat{G}(t-\tau)\alpha(T(\tau)) \frac{\partial T(\tau)}{\partial \tau} d\tau \\ &= \int_0^\xi G(\xi-\xi')\alpha(T(\xi')) \frac{\partial T(\xi')}{\partial \xi'} d\xi'\end{aligned} \tag{56}$$

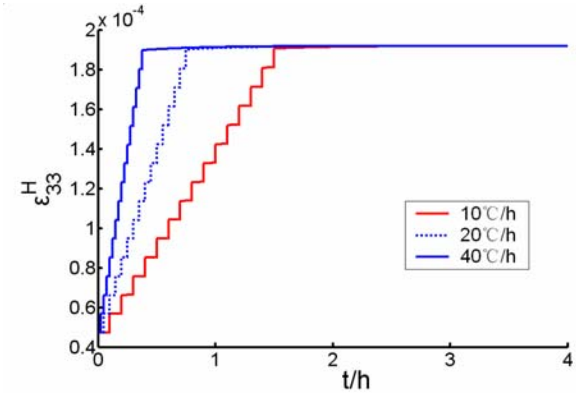

Fig.7 Strain curve under various rate of temperature

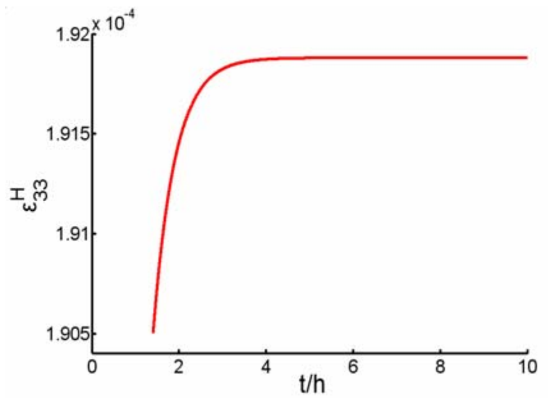

Fig.8 Strain curve under the last step of temperature

4 Summary and Conclusion

In this paper we have developed a novel multi-scale method for analyzing the viscoelastic property of unidirectional composite materials, and have investigated the character of the viscoelastic relaxation law under varying temperature. A homogenization-based method for predicting the effective viscoelastic relaxation modulus considering the temperature-dependency of properties was given.

The numerical results show that, when the temperature increased uniformly step by step, the relaxation rate is so high that the transient process is very quick. That is, the time effect of the relaxation is small, so the coefficient of the thermal expansion may be given by the transient thermal deformation.

4 Acknowledgements

This research is supported in part by the NSFC through the Grant No.s (10572030, 10332010 and 10421202), the National Basic Research Program of China. The financial contributions are gratefully acknowledged.

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